SB-3530

M. A. / M. Sc. (Part - II) Examination

March / April – 2011

Mathematics : Paper - 501

(Operator Theory) (New Course)

Time : **3** Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :

☛ **M. A. / M. Sc. (Part - 2)**

Name of the Subject :

☛ **Mathematics : Paper - 501 (New)**

☛ Subject Code No. : **3 5 3 0** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Answer all questions.

(3) Each question carries **14** marks.

(4) Figures to the right indicate marks of the question.

(5) Follow usual notations.

1 (a) Let X and Y be Banach spaces and $T : D(T) \to Y$ a closed linear operator, where $D(\mathrm{T}) \subset X$. Then prove that if $D(T)$ is closed in X, the operator T is bounded. **6**

(b) Attempt any **two** : **8**

(i) Show that every Hilbert space is reflexive.

(ii) Show that in a finite dimensional normed space the distinction between weak convergence and strong convergence disappears.

(iii) If (x_n) in a Banach space X is such that $\left(f\left(x_n\right)\right)$ is bounded for all $f \in X'$, then show that $\left(\|x_n\|\right)$ is bounded.

2 (a) Prove that the resolvent *p(T)* of a bounded linear operator T on a complex Banach space X is open. **6**

(b) Attempt any **two** : **8**

(i) Prove that the spectrum of a bounded linear operator $T : X \to X$ on a complex Banach space X is compact and lies in the disk given by $|\lambda| \le \|T\|$.

(ii) Prove that if $X \neq \{0\}$ is a complex Banach space and $T \in B(X, X)$ then $\sigma(T) \neq \phi$.

(iii) If $S, T \in B(X, X)$, show that for any $\lambda \in \rho(S) \cap \rho(T), R_\lambda(S) - R_\lambda(T) = R_\lambda(S)(T - S)R_\lambda(T)$.

3 (a) Show that if a compact linear operator on a normed space has infinitely many eigenvalues, then these eigenvalues can be arranged in a sequence converging to zero. **6**

(b) Attempt any **two** : **8**

(i) Show that compact linear operators from a normed space X into a normed space Y constitute a subspace $C(X,Y)$ of $B(X,Y)$.

(ii) Prove that the range of a compact linear operator defined on a normed space X into a normed space Y is separable.

(iii) Prove that if a subset B of a metric space X is relatively compact, it is totally bounded.

4 (a) Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then prove that : **6**

(i) All the eigenvalues of T(if they exists) are real.

(ii) Eigen vectors corresponding to different eigenvalues of T are orthogonal.

(b) Attempt any **two** : **8**

(i) Let P_1 and P_2 be projections on a Hilbert space H. Then show that $P_2 - P_1$ is a projection on $H \Leftrightarrow Y_1 \subset Y_2$ where $Y_j = P_j(H)$.

(ii) Prove that a bounded linear operator $P: H \to H$ on a Hilbert space H is a projection $\Leftrightarrow P$ is self-adjoint and idempotent.

(iii) Prove that the spectrum of a bounded self-adjoint linear operator $T: H \to H$ on a complex Hilbert space H lies in the closed interval $[m, M]$ on the real axis, where $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x \rangle$.

5 (a) Let $T: D(T) \to H$ be a symmetric linear operator defined denesly in a complex Hilbert space H. Then prove that $\left(\overline{H}\right)^* = T^*$. **6**

(b) Attempt any **two**: **8**

(i) For linear operators $T : D(T) \to H$ and $S : D(S) \to H$ defined densely in a complex Hilbert space H, Prove that $S \subset T \Rightarrow T^* \subset S^*$.

(ii) Prove that the spectrum $\sigma(T)$ of a self-adjoint linear operator $T : D(T) \to H$ is real, where H is a complex Hilbert space and $D(T)$ is dense in H.

(iii) Prove that a densely defined linear operator T in a complex Hilbert space H is symmetric. if and only if $T \subset T^*$.